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Algorithms, Probability, and Computing **Solutions KW40** **HS23**

Solution 1

(a) The standard BasicMinCut algorithm can be implemented using an array containing all the degrees of the vertices currently in the graph — for the purpose of efficiently selecting an edge u.a.r. for contraction. The only adaptation needed now is that after each contraction, we scan this array and maintain a global minimal degree ever seen. It is clear that this can be done in linear time per step and thus in $\mathcal{O}(n^2)$ time in total.

(b) We know from the lecture that contractions can only increase the size of a minimum cut but never decrease it. Since the edges incident to any one vertex always *form* a cut, each of the numbers that we could report in this algorithm corresponds to some cut in the original graph, which readily implies the claim.

(c) For $n \leq 2$, the claim is empty. Let us now look at $n > 2$, let us fix a graph G of size n and a cut C of size $\mu(G)$. What is the probability that the event we are looking for occurs? There are two cases. Either, G contains a vertex of degree less than $(1+\alpha)\mu(G)$. In that case, no matter what the further recursion would yield, we will always return a number at most that degree and thus the probability is 1. Or, all vertices in G have degree at least $(1+\alpha)\mu(G)$. Then, there are at least $(1+\alpha)\mu(G) \cdot \frac{n}{2}$ edges in the graph and the probability that we contract one from C is thus bounded by $\frac{2}{(1+\alpha)n}$. But then with the complement of this probability, the new graph G/e will still have a cut of size $\mu(G)$ and by induction, the claim follows.

As for the calculation (which was not required), since $p_\alpha(2) = 1$, we just compute (for

n sufficiently large)

$$
\begin{aligned}
p_\alpha(n) &\geq \prod_{i=3}^{n} \left(1 - \frac{2}{(1+\alpha)i}\right) \\
&= \exp\left(\sum_{i=3}^{n} \ln\left(1 - \frac{2}{(1+\alpha)i}\right)\right) \\
&\overset{(1)}{\geq} \exp\left(\sum_{i=3}^{n} \left(-\frac{2}{(1+\alpha)i} - \left(\frac{2}{(1+\alpha)i}\right)^2\right)\right) \\
&\overset{(2)}{\geq} \exp\left(-\frac{2}{1+\alpha}\left(H_n - \frac{3}{2}\right) - \frac{2\pi^2}{6}\right) \\
&\overset{(3)}{\geq} \exp\left(-\frac{2}{1+\alpha}\ln n - \frac{2\pi^2}{6}\right) \\
&= \Omega(n^{\frac{2}{1+\alpha}}),
\end{aligned}
$$

where (1) uses the inequality $1 - x \geq e^{-x-x^2}$ for $x \leq 0.68$ and $\frac{2}{(1+\alpha)i} < 2/3 < 0.68$ for all $\alpha > 0$ and $i \geq 3$; where (2) uses that $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$ and $\frac{2}{1+\alpha} < 2$ for $\alpha > 0$; and where (3) uses $H_n - 3/2 < \ln n$ (the difference between $\ln n$ and H_n approaches the Euler-Mascheroni constant γ, which is about 0.577).

Solution 2

Given a graph $G = (V, E)$, let N be the number of minimum cuts in G. We want to show that $N \leq \binom{n}{2}$ where $n := |V|$.

Let $C_1, \ldots, C_N$ be the minimum cuts in G. Then, we know that for each $i \in \{1..N\}$

$$\Pr[C_i \text{ is found by Karger's algorithm } \textsc{BasicMinCut}(G)] \geq \frac{1}{\binom{n}{2}}.$$

Now, we observe that for each two distinct indices $i, j \in \{1..N\}$ the events "C_i is found by $\textsc{BasicMinCut}(G)$" and "C_j is found by $\textsc{BasicMinCut}(G)$" are disjoint (i.e. they never happen at the same time). To see this, consider the graph obtained at the termination of $\textsc{BasicMinCut}(G)$. It has only two vertices and these vertices (together with the "contraction history") uniquely determine a partition of the vertex set V. So we cannot get two different minimum cuts from one execution of the algorithm. Therefore, it follows that

$$
\begin{aligned}
&\Pr[\text{a minimum cut is found by } \textsc{BasicMinCut}(G)] \\
&= \sum_{i=1}^{N} \Pr[C_i \text{ is found by } \textsc{BasicMinCut}(G)] \geq \frac{N}{\binom{n}{2}}.
\end{aligned}
$$

Since Pr[a minimum cut is found by BasicMinCut(G)] ≤ 1 (because it is a probability!), we obtain $N \leq \binom{n}{2}$.

Solution 3

(a) Observe that there are three (potentially empty) sets of edges $e = \{u, v\}$ that are important in this scenario:

$$
\begin{aligned}
E_1 &:= \left\{e = \{u, v\} \mid u \in A \cap B \text{ and } v \in V \setminus (A \cup B)\right\} \\
E_2 &:= \left\{e = \{u, v\} \mid u \in (A \setminus B) \cup (B \setminus A) \text{ and } v \in V \setminus (A \cup B)\right\} \\
E_3 &:= \left\{e = \{u, v\} \mid u \in A \cap B \text{ and } v \in (A \setminus B) \cup (B \setminus A)\right\}.
\end{aligned}
$$

It is easy to see that $f(A \cap B) + f(A \cup B) = |E_1| + |E_3| + |E_1| + |E_2|$, while $f(A) + f(B) \geq 2|E_1| + |E_2| + |E_3|$, the last inequality holding because some edges between $A \setminus B$ and $B \setminus A$ may appear both in $f(A)$ and $f(B)$.

(b) Let k be the size of a minimum cut. Then by (a) we get $f(A \cap B) + f(A \cup B) \leq f(A) + f(B) = k + k$, which together with $f(C) \geq k$ for any set $C \neq \emptyset, V$, implies $f(A \cap B) = f(A \cup B) = k$.

(c) Suppose towards a contradiction that $S \neq S'$, with $S, S' \subset V$ and $s \in S \cap S'$ are such that $C(S)$ and $C(S')$ are both minimum cuts and $|S| = |S'|$ is minimal. Note that as $t \notin S \cup S'$ because they are both cuts, we must have $S \cup S' \neq V$, and so we can in a similar way to part (b) prove that $S \cap S'$ is a minimum cut with $|S \cap S'| < |S| = |S'|$, a contradiction.

Solution 4

(a) Since G is connected, there are at least $n - 1$ edges. If there are at least n edges, then $\Pr[\mu(G) \neq \mu(G/e)] \leq \frac{1}{n}$ since in the worst case the minimum cut is unique and the probability of contracting a given edge is at most the claimed bound.

If there are only $n - 1$ edges the graph is a tree. Then contracting any edge keeps the graph a tree with 1 fewer vertex which means that $\Pr[\mu(G) \neq \mu(G/e)] = 0 \leq \frac{1}{n}$.

(b) BasicMinCut will succeed if it never contracts a given edge in the cut of size 1. This happens with probability at least

$$
\begin{aligned}
&\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n-1}\right) \cdots \left(1 - \frac{1}{3}\right) \\
&= \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1}\right) \cdots \left(\frac{2}{3}\right) = \frac{2}{n}.
\end{aligned}
$$

(c) A contraction of an edge changes the degree of at most 2 vertices. Therefore there is still at least one vertex of degree $k \Rightarrow$ minimum cut is still of size k.

(d) If there are 3 or more vertices of degree k we are done by (c). If there are 0 or 1 vertices of degree k, then the number of edges is at least $((n-1)(k+1)+k)/2 = (n(k+1)-1)/2$ so that when we fix a minimum cut of size k, the probability of contracting one of the edges of the minimum cut is at most

$$\frac{k}{(n(k+1)-1)/2} = \frac{2k}{n(k+1)-1}.$$

It remains to consider the case that there are two vertices of degree k. If these two vertices are not adjacent, then there are two disjoint minimum cuts which means that $\Pr[\mu(G) \neq \mu(G/e)] = 0$. If they are adjacent, then the minimum cut may only change if we contract one of the at most $k-1$ edges between the two vertices (they can't be connected via k edges since the graph is connected and $n \geq 3$). Since there are at least $(2k + (n-2)(k+1))/2$ edges, the failure probability is at most

$$\frac{2(k-1)}{2k+(n-2)(k+1)} \leq \frac{2k-2}{n(k+1)-2} \leq \frac{2k-1}{n(k+1)-1} \leq \frac{2k}{2n(k+1)-1}.$$

The second to last inequality holds because $a := 2k-1 \leq n(k+1)-1 =: b$ and then

$$\frac{a-1}{b-1} \leq \frac{a}{b} \Leftrightarrow -b \leq -a \Leftrightarrow a \leq b.$$