Advanced Graph Algorithms and Optimization Spring 2020

Course Introduction

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The exercises for this week will not count toward your grade, but you are highly encouraged to solve them all.

Exercise 1.

Let us define the α-*sub-level set* of a function $f : \mathbb{R}^n \to \mathbb{R}$ to be the set $S_\alpha \stackrel{\text{def}}{=} \{\boldsymbol{x} : f(\boldsymbol{x}) \leq \alpha\}$.

(i) Prove that if a function f is convex, then all its sub-level sets are convex sets.

(ii) Is it true that a function whose sub-levels sets are all convex is necessarily convex?

Exercise 2.

Recall the definition of Laplacian $\boldsymbol{L} = \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top$.

(i) We can also define Laplacian as $\boldsymbol{L} \stackrel{\text{def}}{=} \boldsymbol{D} - \boldsymbol{A}$, where $\boldsymbol{A}$ is the weighted adjacency matrix, i.e. $\boldsymbol{A}(u, v) = 1/\boldsymbol{r}(u, v)$, and $\boldsymbol{D} \stackrel{\text{def}}{=} \operatorname{diag}_{v\in V} \boldsymbol{w}(v)$ for $\boldsymbol{w}(v) := \sum_{(u,v)\in E} 1/\boldsymbol{r}(u, v)$. Prove that these two definitions are equivalent.

(ii) Given a function on the vertices, $\boldsymbol{x} \in \mathbb{R}^V$, the Laplacian quadratic form is

$$\boldsymbol{x}^\top \boldsymbol{L}\boldsymbol{x} = \sum_{(u,v)\in E} \frac{(\boldsymbol{x}(u) - \boldsymbol{x}(v))^2}{\boldsymbol{r}(u, v)}.$$

Prove the above equality and building on that, show that $\boldsymbol{L}$ is positive semi-definite.

(iii) What is the kernel of $\boldsymbol{L}$, which is denoted by $\operatorname{Ker}(\boldsymbol{L})$?

Exercise 3.

(i) Prove that for a matrix $\boldsymbol{A}$ we have $\operatorname{im}(\boldsymbol{A}) = \ker(\boldsymbol{A}^\top)^\perp$, where $\operatorname{im}(\boldsymbol{A})$ denotes the image of $\boldsymbol{A}$ and $\ker(\boldsymbol{A}^\top)^\perp$ is the orthogonal complement to $\ker(\boldsymbol{A}^\top)$.

(ii) Building on part (i), prove that in our flow problem, when the graph is connected, an electrical flow $\boldsymbol{f}$ routing $\boldsymbol{d}$ exists if and only if $\boldsymbol{1}^\top \boldsymbol{d} = 0$.

Exercise 4.

Define the gradient of a multivariate function $f : S \to \mathbb{R}$ for $S \subseteq \mathbb{R}^n$. Then, prove that the system of linear equations $\boldsymbol{L}\boldsymbol{x} = \boldsymbol{d}$ is the same as the system obtained by setting the gradient with respect to $\boldsymbol{x}$ of the function $c(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{L}\boldsymbol{x} - \boldsymbol{x}^\top \boldsymbol{d}$ equal to zero.

Exercise 5.

(i) Recall that the electrical flow and voltages satisfy $\boldsymbol{f}^* = \boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{x}^*$ and $\boldsymbol{B}\boldsymbol{f}^* = \boldsymbol{d}$. Prove that $(\boldsymbol{f}^*)^\top \boldsymbol{R}\boldsymbol{f}^* = (\boldsymbol{x}^*)^\top \boldsymbol{L}\boldsymbol{x}^*$.

(ii) Conclude that

$$\max_{\boldsymbol{x}\in\mathbb{R}^V} \boldsymbol{x}^\top \boldsymbol{d} - \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{L}\boldsymbol{x} = \min_{\boldsymbol{f}\in\mathbb{R}^E} \frac{1}{2}\sum_e \boldsymbol{r}(e)\boldsymbol{f}(e)^2 \quad \text{s.t. } \boldsymbol{B}\boldsymbol{f} = \boldsymbol{d}.$$