Advanced Graph Algorithms and Optimization Spring 2020

Some Basic Optimization, Convex Geometry, and Linear Algebra

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The exercises for this week will not count toward your grade, but you are highly encouraged to solve them all.

Exercise 1.

Recall that the following theorem gives us a sufficient (though not necessary) condition for optimality.

Theorem (Extreme Value Theorem)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and $\mathcal{F} \subseteq \mathbb{R}^n$ be nonempty, bounded, and closed. Then, the optimization problem $\min f(\boldsymbol{x}) : \boldsymbol{x} \in \mathcal{F}$ has an optimal solution.

Prove the above theorem. You might use the following two theorems.

Theorem (Bolzano-Weierstrass)**.** Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Theorem (Boundedness Theorem)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and $\mathcal{F} \subseteq \mathbb{R}^n$ be nonempty, bounded, and closed. Then f is bounded on $\mathcal{F}$.

Exercise 2.

Prove Taylor's Theorem.

Theorem (Taylor's Theorem, multivariate first-order remainder form)**.** If $f : S \to \mathbb{R}$ is continuously differentiable over $[\boldsymbol{x}, \boldsymbol{y}]$, then for some $\boldsymbol{z} \in [\boldsymbol{x}, \boldsymbol{y}]$,

$$f(\boldsymbol{y}) = f(\boldsymbol{x}) + \boldsymbol{\nabla} f(\boldsymbol{z})^\top (\boldsymbol{y} - \boldsymbol{x}).$$

Exercise 3.

Let $f_1(\boldsymbol{x}), \cdots, f_k(\boldsymbol{x})$ be a set of convex functions with the same domain and define

$$f(\boldsymbol{x}) \stackrel{\text{def}}{=} \max_{1 \le i \le k} f_i(\boldsymbol{x}).$$

Prove that $f(\boldsymbol{x})$ is convex.

Exercise 4.

Assume that $f(x, y)$ is a convex function and S is a convex non-empty set. Prove that

$$g(x) = \inf_{y \in S} f(x, y)$$

is convex, provided $g(x) > -\infty$ for all x.

Exercise 5.

Prove that if a matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is symmetric, then $\|\boldsymbol{A}\| = \max(|\lambda_{\max}(\boldsymbol{A})|, |\lambda_{\min}(\boldsymbol{A})|)$ and give an example of a non-symmetric matrix for which this is not true.

Exercise 6.

For each function below, determine whether it is convex or not.

1. $f(x) = |x|^6$ on $x \in \mathbb{R}$
2. $f(x) = \exp(x)$ on $x \in (0, \infty)$
3. $f(x, y) = \sqrt{x + y}$ on $(x, y) \in (0, 1) \times (0, 1)$
4. $f(x, y) = xy$ on $(x, y) \in (-1, 1) \times (-1, 1)$