Advanced Graph Algorithms and Optimization Spring 2020

Convexity and Second Derivatives, Gradient Descent and Acceleration

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The exercises for this week will not count toward your grade, but you are highly encouraged to solve them all.

Exercise 1.

Consider a twice differentiable function $f : S \to \mathbb{R}$, where $S \subset \mathbb{R}^n$ is a convex open set. Prove that f is β-gradient Lipschitz if and only if for all $\boldsymbol{x} \in S$ (except a measure zero set), $\|\lambda_{\max}(\boldsymbol{H}_f(\boldsymbol{x}))\| \leq \beta$.

Exercise 2.

Prove that when running Gradient Descent, $\|\boldsymbol{x}_i - \boldsymbol{x}^*\|_2 \leq \|\boldsymbol{x}_0 - \boldsymbol{x}^*\|_2$ for all i.

Exercise 3.

Prove the following theorem.

Theorem. Let $f : \mathbb{R}^n \to \mathbb{R}$ be an β-gradient Lipschitz, convex function. Let $\boldsymbol{x}_0$ be a given starting point, and let $\boldsymbol{x}^* \in \arg\min_{\boldsymbol{x}\in\mathbb{R}^n} f(\boldsymbol{x})$ be a minimizer of f. The Gradient Descent algorithm given by

$$\boldsymbol{x}_{i+1} = \boldsymbol{x}_i - \frac{1}{\beta}\boldsymbol{\nabla} f(\boldsymbol{x}_i)$$

ensures that the kth iterate satisfies

$$f(\boldsymbol{x}_k) - f(\boldsymbol{x}^*) \leq \frac{2\beta\,\|\boldsymbol{x}_0 - \boldsymbol{x}^*\|_2^2}{k+1}.$$

Hint: do an induction on $1/gap_i$.

Exercise 4.

1. For each of the following functions answer these questions:
   - Is the function convex?
   - Is the function β-gradient Lipschitz for some β?
   - If the function is β-gradient Lipschitz give an upper bound on β – the bound should be within a factor 4 of the true value.

   (a) $f(x) = |x|^{1.5}$ on $x \in \mathbb{R}$

(b) $f(x) = \exp(x)$ on $x \in \mathbb{R}$

(c) $f(x) = \exp(x)$ on $x \in (-1, 1)$

(d) $f(x, y) = \sqrt{x + y}$ on $(x, y) \in (0, 1) \times (0, 1)$.

(e) $f(x, y) = \sqrt{x + y}$ on $(x, y) \in (1/2, 1) \times (1/2, 1)$.

(f) $f(x, y) = \sqrt{x^2 + y^2}$ on $(x, y) \in \mathbb{R}^2$.