Introduction to Spectral Graph Theory

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The exercises for this week will not count toward your grade, but you are highly encouraged to solve them all.

Exercise 1.

Prove that

1. $\boldsymbol{v}_0 = \boldsymbol{x}_0 - \frac{a_0}{\sigma}\boldsymbol{\nabla} f(\boldsymbol{x}_0)$
2. $L_0 = f(\boldsymbol{x}_0) - \frac{a_0}{2\sigma}\left\|\boldsymbol{\nabla} f(\boldsymbol{x}_0)\right\|_2^2 - \frac{\sigma}{2a_0}\left\|\boldsymbol{x}^* - \boldsymbol{x}_0\right\|_2^2$.

Exercise 2.

Prove that

1. $m_i(\boldsymbol{v}) = m_i(\boldsymbol{v}_i) + \frac{\sigma}{2}\left\|\boldsymbol{v} - \boldsymbol{v}_i\right\|_2^2$
2. $m_{i+1}(\boldsymbol{v}) = m_i(\boldsymbol{v}) + a_{i+1} f(\boldsymbol{x}_{i+1}) + \langle a_{i+1}\boldsymbol{\nabla} f(\boldsymbol{x}_{i+1}), \boldsymbol{v} - \boldsymbol{x}_{i+1}\rangle$
3. $\boldsymbol{v}_{i+1} = \boldsymbol{v}_i - \frac{a_0}{\sigma}\boldsymbol{\nabla} f(\boldsymbol{x}_0)$.

Exercise 3.

1. Assume that $S \subseteq \mathbb{R}^n$ is a convex set and that the function $f : S \to \mathbb{R}$ is convex. Suppose that $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n \in S$ and $\theta_1, \cdots, \theta_n \geq 0$ with $\theta_1 + \cdots + \theta_n = 1$. Prove that
$$f(\theta_1\boldsymbol{x}_1 + \cdots + \theta_n\boldsymbol{x}_n) \leq \theta_1 f(\boldsymbol{x}_1) + \cdots + \theta_n f(\boldsymbol{x}_n).$$
Remark. This is typically known as Jensen's inequality and can be extended to infinite sums. If $\mathcal{D}$ is a probability distribution on S, and $\boldsymbol{X} \sim \mathcal{D}$, then
$$f(\mathbb{E}[\boldsymbol{X}]) \leq \mathbb{E}[f(\boldsymbol{X})]$$
whenever both integrals are finite.
2. Prove that
$$\left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^n x_i.$$
3. Prove that
$$\frac{1}{\frac{1}{n}\sum_{i=1}^n \frac{1}{x_i}} \leq \left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}}.$$