Spectral Graph Theory

Rasmus Kyng *Problem Set 5 — Wednesday, April 8*

The exercises for this week will not count toward your grade, but you are highly encouraged to solve t hem all. The solution is due on **Sunday, April 19 by 12:00 noon**.

Througout these exercises, we will use the following notation:

- S^n is the set of symmetric real matrices $n \times n$ matrices.
- S^n_+ is the set of positive semi-definite $n \times n$ matrices.
- S^n_{++} is the set of positive definite $n \times n$ matrices.

Whenever we say a matrix is positive semi-definite or positive definite, we require it to be real and symmetric.

Exercise 1.

1. Show that there exist two matrices $\boldsymbol{A}, \boldsymbol{B} \in S^n_{++}$ such that $\boldsymbol{A} \preceq \boldsymbol{B}$ but $\boldsymbol{A}^2 \not\preceq \boldsymbol{B}^2$.
2. Let $\boldsymbol{A}, \boldsymbol{B} \in S^n_{++}$, and assume $\boldsymbol{A} \preceq \boldsymbol{B}$. Prove that $\boldsymbol{B}^{-1} \preceq \boldsymbol{A}^{-1}$.

   Hint: It might help to first prove that for a matrix $\boldsymbol{C} \in \mathbb{R}^{n\times n}$, we have $\boldsymbol{C}\boldsymbol{A}\boldsymbol{C}^\top \preceq \boldsymbol{C}\boldsymbol{B}\boldsymbol{C}^\top$.

Exercise 2.

Let $\boldsymbol{M} = \boldsymbol{X}\boldsymbol{Y}\boldsymbol{X}^\top$ for some $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{n\times n}$, where $\boldsymbol{X}$ is invertible and $\boldsymbol{M}$ is symmetric. Furthermore, consider the spectral decomposition of $\boldsymbol{M} = \sum_{i=1}^n \lambda_i \boldsymbol{v}_i \boldsymbol{v}_i^\top$. Then, we define $\boldsymbol{\Pi}_{\boldsymbol{M}} = \sum_{i,\lambda_i \neq 0} \boldsymbol{v}_i \boldsymbol{v}_i^\top$. $\boldsymbol{\Pi}_{\boldsymbol{M}}$ is the orthogonal projection onto the image of $\boldsymbol{M}$: It has the property that for $\boldsymbol{v} \in \operatorname{im}(\boldsymbol{M})$, $\boldsymbol{\Pi}_{\boldsymbol{M}}\boldsymbol{v} = \boldsymbol{v}$ and for $\boldsymbol{v} \in \ker(\boldsymbol{M})$, $\boldsymbol{\Pi}_{\boldsymbol{M}}\boldsymbol{v} = \boldsymbol{0}$.

Prove that

$$\boldsymbol{Z} = \boldsymbol{\Pi}_{\boldsymbol{M}}(\boldsymbol{X}^\top)^{-1}\boldsymbol{Y}^+\boldsymbol{X}^{-1}\boldsymbol{\Pi}_{\boldsymbol{M}}$$

is the pseudoinverse of $\boldsymbol{M}$.

For a matrix $\boldsymbol{Z}$ to be the pseudoinverse of a symmetric matrix $\boldsymbol{M}$, you need to show that

1. $\boldsymbol{Z}^\top = \boldsymbol{Z}$.
2. $\boldsymbol{Z}\boldsymbol{v} = \boldsymbol{0}$ for $\boldsymbol{v} \in \ker(\boldsymbol{M})$.
3. $\boldsymbol{M}\boldsymbol{Z}\boldsymbol{v} = \boldsymbol{v}$ for $\boldsymbol{v} \in \ker(\boldsymbol{M})^\perp$.

Exercise 3.

In this exercise, we want you to complete the proof of Theorem 2.3 in Lecture 8. Refer to the lectures notes for definitions of the terms used here.

1. Prove that Equation (3) is satisfied, i.e. that for all edges $e \in E$ we have $\|\boldsymbol{X}_e\| \leq \frac{1}{\alpha}$.
2. Prove that Equation (4) is satisfied, i.e. that $\left\|\sum_e \mathbb{E}\left[\boldsymbol{X}_e^2\right]\right\| \leq \frac{1}{\alpha}$.
3. Explain how we can use a scalar Chernoff bound to prove that $\left|\tilde{E}\right| \leq O(\epsilon^{-2}\log(n/\delta)n)$ with probability at least $1-\delta/2$. You may pick any constant that suits you to establish the $O(\cdot)$ bound.

Exercise 4.

Consider $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{v}, \boldsymbol{u} \in \mathbb{R}^n$.

1. Assume that $\boldsymbol{I} + \boldsymbol{u}\boldsymbol{v}^\top$ is invertible. Determine c such that

$$\left(\boldsymbol{I} + \boldsymbol{u}\boldsymbol{v}^\top\right)^{-1} = \boldsymbol{I} - \frac{\boldsymbol{u}\boldsymbol{v}^\top}{c}.$$

2. Assume that both $\boldsymbol{A}$ and $\boldsymbol{A} + \boldsymbol{u}\boldsymbol{v}^\top$ are invertible. Prove that

$$\left(\boldsymbol{A} + \boldsymbol{u}\boldsymbol{v}^\top\right)^{-1} = \boldsymbol{A}^{-1} - \frac{\boldsymbol{A}^{-1}\boldsymbol{u}\boldsymbol{v}^\top\boldsymbol{A}^{-1}}{1 + \boldsymbol{v}^\top\boldsymbol{A}^{-1}\boldsymbol{u}}.$$

Hint: You might use that $(\boldsymbol{B}\boldsymbol{C})^{-1} = \boldsymbol{C}^{-1}\boldsymbol{B}^{-1}$ for two invertible matrices $\boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}^{n\times n}$.

Exercise 5.

Consider a matrix function $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$. For $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{n\times n}$, we define

$$Df\left(\boldsymbol{X}\right)\left[\boldsymbol{Y}\right] = \left.\frac{\partial}{\partial t}\right|_{t=0} f\left(\boldsymbol{X} + t\boldsymbol{Y}\right).$$

Remark. Note that if we think of $\boldsymbol{X}$ and $\boldsymbol{Y}$ each as a vector of numbers, then this is the (matrix-valued) directional derivative of f at $\boldsymbol{X}$ in the direction of $\boldsymbol{Y}$.

Consider $f(\boldsymbol{X}) = \boldsymbol{X}^{-1}$ for an invertible matrix $\boldsymbol{X} \in \mathbb{R}^{n\times n}$. Prove that

$$Df(\boldsymbol{X})[\boldsymbol{Y}] = -\boldsymbol{X}^{-1}\boldsymbol{Y}\boldsymbol{X}^{-1}.$$

Hint: You might need to use Exercise 4.