Advanced Graph Algorithms and Optimization	Spring 2020

Classical Algorithms for Maximum Flow

Rasmus Kyng — *Problem Set 7 — Wednesday, April 29*

The exercises for this week will not count toward your grade, but you are highly encouraged to solve them all. The solution is due on **Monday, May 4 by 12:00 noon**.

In Exercises 1 and 2, we'll explore algorithms finding a maximum flow on a directed graph with edge capacities.

Consider directed graph $G = (V, E, \boldsymbol{c})$ with arbitrary capacities $\boldsymbol{c} \geq \boldsymbol{0}$.

Let $\boldsymbol{B} \in \mathbb{R}^{E \times V}$ be the edge vertex incidence matrix of the graph, i.e. if $e \in E$ and $(u, v) = e$ then $\boldsymbol{B}(e, u) = 1$ and $\boldsymbol{B}(e, v) = -1$.

We let $\boldsymbol{\chi}_v \in \mathbb{R}^V$ denote the indicator of vertex v, i.e. $\boldsymbol{\chi}_v(v) = 1$ and $\boldsymbol{\chi}_v(u) = 0$ for $u \neq v$.

We let $s \in V$ denote the flow "source" and $t \in V$ the flow "sink".

The maximum flow problem is given by

$$\begin{aligned} \max_{\boldsymbol{f} \in \mathbb{R}^E, F \geq 0} \quad & F \\ \text{s.t. } \boldsymbol{B}\boldsymbol{f} &= F(-\boldsymbol{\chi}_s + \boldsymbol{\chi}_t) \\ \boldsymbol{0} \leq \boldsymbol{f} &\leq \boldsymbol{c} \end{aligned}$$

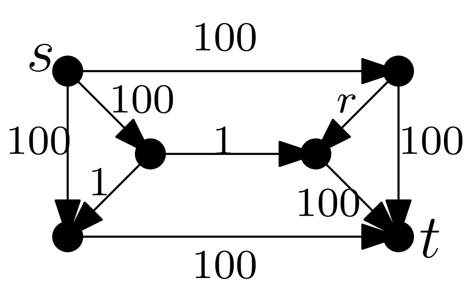
In the context of a given maximum flow problem, for a flow $\boldsymbol{f}$ satisfying $\boldsymbol{B}\boldsymbol{f} = F(-\boldsymbol{\chi}_s + \boldsymbol{\chi}_t)$, we define $\mathrm{val}(\boldsymbol{f}) = F$.

Let $\boldsymbol{f}^*$ denote a feasible flow maximizing F, so that the maximum attainable flow value F is $\mathrm{val}(\boldsymbol{f}^*)$.

Exercise 1: Convergence of Ford-Fulkerson

Show that the Ford-Fulkerson algorithm may not terminate; moreover, it may converge a value not equal to the value of the maximum flow.

Hint: You might use the graph below with the given capacities, where $r = \frac{\sqrt{5}-1}{2}$ *(which implies that* $r^2 = 1 - r$*).*



Exercise 2: Iterative Refinement for Maximum Flow

Suppose we have an algorithm FlowRefine, which given a maximum flow instance $G = (V, E, \boldsymbol{c})$ with source $s \in V$ and sink $t \in V$ returns a feasible s-t flow $\tilde{\boldsymbol{f}}$, i.e. $\boldsymbol{B}\tilde{\boldsymbol{f}} = F(-\boldsymbol{\chi}_s + \boldsymbol{\chi}_t)$ for some F, and $\boldsymbol{0} \leq \tilde{\boldsymbol{f}} \leq \boldsymbol{c}$, and $\tilde{\boldsymbol{f}}$ is guaranteed to route at least half the maximum flow, i.e. $F = \mathrm{val}(\tilde{\boldsymbol{f}}) \geq 0.5\,\mathrm{val}(\boldsymbol{f}^*)$.

Suppose that the running time of FlowRefine is $O(|E|^c)$ for some constant $c \geq 1$.

Explain how we can use FlowRefine to find a flow $\hat{\boldsymbol{f}}$ that routes at least $(1-\epsilon)\,\mathrm{val}(\boldsymbol{f}^*)$ in time $O(|E|^c \log(1/\epsilon))$.

Exercise 3: A Scalar Martingale Theorem

The following is a concentration theorem for scalar martingales, that can be quite useful when the "pseudo-variance" of the martingale (see below) is easy to bound. This often turns out to be the case.

Theorem. Suppose the random variables $X_1, X_2, \ldots, X_k \in \mathbb{R}$ form a scalar martingale difference sequence, i.e.

$$\mathbb{E}\left[X_i \mid X_1, \ldots, X_{i-1}\right] = 0$$

Suppose also that $|X_i| \leq 1$ always. Define the "pseudo-variance"

$$W_i = \sum_{j=1}^{i} \mathbb{E}\left[X_j^2 \mid X_1, \ldots, X_{j-1}\right]$$

Then

$$\Pr\left[\left|\sum_{i=1}^{k} X_i\right| \geq t \text{ and } W_k \leq \sigma^2\right] \leq C_2 \exp\left(-C_1 \frac{t^2}{Rt + \sigma^2}\right)$$

The exercise is to prove this theorem for some fixed constants C_1 and C_2, e.g. $C_1 = 1/10$ and $C_2 = 100$ or other constants you can make work. They should not depend on R or σ^2 or t.

Hint: you may find it useful to evaluate the mean-exponential $\mathbb{E}\left[\exp(aX - bW)\right]$ for some parameters a and b.

Remark. Note that W_i is a random variable! But it is fixed, i.e. *not* random, conditional on $X_1, \ldots, X_{i-1}$. When we use this theorem, it has to be the case that we know something about the martingale that helps us show the probability $\Pr[W_k > \sigma^2]$ is small. Then to bound $\Pr\left[\left|\sum_{i=1}^{k} X_i\right| \geq t\right]$, we use a union bound:

$$\Pr\left[\left|\sum_{i=1}^{k} X_i\right| \geq t\right] \leq \Pr\left[\left|\sum_{i=1}^{k} X_i\right| \geq t \text{ and } W_k \leq \sigma^2\right] + \Pr[W_k > \sigma^2].$$

The above theorem is used to bound the first term, while we need some other way to bound the second term.